Treatment of non-linearity in mesoscale modeling

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In the Navier-Stokes equations NSE, the velocity enters non-linearly and the viscosity enters linearly. One substitutes the NSE with *stochastic Langevin equations in which the velocity enters linearly* while *the non-linearity is now in the viscosity which becomes a turbulent viscosity*. The same procedure applies to the tracer equation with the non-linearity represented by a *turbulent diffusivity*. In many cases, the linear nature of the Langevin equations yields equations that can be solved analytically. *The challenge is to prove that the fluxes (second-order correlations) thus obtained reproduce those obtained from direct numerical simulations (DNS) and large eddy simulations (LES) of the original NSE, as well as data from laboratory, in situ measurements and satellites*.

Kraichnan (1971) and Leith (1971) pioneered this methodology and Herring and Kraichnan (1971) presented a review of Langevin-type equations for different turbulent diffusivities and compared the performance of these models vis a′ vis each other, numerical simulations and laboratory data. Kraichnan (1975) suggested Langevin-type equations to construct a SGS (sub-grid scale) models for use in LES, a suggestion later pursued by Bertoglio (1985) to study homogeneous shear turbulence. The Langevin equation was used by Chasnov (1991) to construct sub-grid scale (SGS) models. He found that the LES results were significantly improved over the eddy-viscosity model to treat SGS. Discussions of the Langevin equations can be found in the books by Lesieur (1990) and McComb (1992).

The above tests were necessary but not sufficient since none of them considered flows of oceanic interest such as shear driven, buoyancy driven, stably stratified, rotating flows etc. Those tests had to be carried out before the Langevin approach could be used to treat the non-linear mesoscale fields. Assessments of the Langevin equations were then carried out with the following results published in *The Physics of Fluids* (1996-1999, electronic copies of the papers are available upon request to the authors): derivation of the Kolmogorov and Batchelor constants and the equation for the turbulent diffusivity in terms of the turbulent viscosity; LES and laboratory data on shear driven flows; laboratory and DNS data on freely decaying turbulence and the behavior of skewness; buoyancy driven flows and laboratory data of the Nusselt number vs. Raleigh number relation; effect of rotation on the kinetic energy spectra and on the energy cascade; DNS data on 2D turbulence; Reynolds stresses for shear driven flows and DNS data and the derivation of the Smagorinsky-Lilly constant that has been widely used to represent unresolved small scale turbulence. About 80 statistics from DNS, LES and laboratory data were reproduced which were considered a validation of the Langevin-based approach.

However, since the method was never discussed in the oceanographic literature, in this report we describe some technical details of the method and how to derive the horizontal tracer flux and the mesoscale diffusivity.

Key features of the Langevin formalism.

In NSE, energy production and dissipation are determined by the linear velocity terms at the largest and smallest scales respectively while the energy transfer from source to sink is governed by the non-linear terms. As direct numerical simulations have shown, the dominant non-linear interactions occur among nearby modes, i.e., in k-space, the energy transfer is local. This means that the energy flows from source to sink in a way similar to a flow of water in a pipe. On the basis of this similarity, Kolmogorov found the celebrated energy spectrum $\sim k^{-5/3}$ for the case of homogeneous isotropic turbulence. The locality approach is also the basis of the 1996-1999 turbulence models cited above. Locality implies that in NSE, the non-linear force acting on a given mode can be divided into contributions from high and low k, i.e. $f_{NL} = f^{>} + f^{<}$ and thus, the NSE can be presented as follows:

$$\frac{\partial}{\partial t} u_i(\mathbf{k}) = f_i(\text{ext}) + f_i^{<}(\text{turb}) + f_i^{>}(\text{turb}) \tag{1}$$

where $f_i(\text{ext})$ is the contribution of the linear terms. As already mentioned, the problem is to find the forms of the two turbulent forces so that the second-order correlations (e.g., heat fluxes obtained by multiplying (1) by the stochastic temperature field and adding it to the equivalent term obtained from the temperature equation analogous to Eq.(1)), are as close as possible to those obtained from the DNS of the original non-linear equations.

The derivation of $f_i^{>}$ was greatly aided by the work of Wyld (1961) who first showed how the Feynman diagram techniques used to treat strong non-linear problems in other contexts, can be applied to treat the velocity field with the following result:

$$f_i^{>}(\text{turb}) = -k^2 \nu_d(k) u_i(k) \tag{2}$$

Eq.(2) has the same form of the kinematic viscosity term in the NSE with the difference that instead of the kinematic viscosity ν, there is the *dynamical viscosity* $\nu_d(k) = \nu + \nu_t(k)$ which is the sum of kinematic plus turbulent viscosity. While ν does not depend on the size of the eddy, $\nu_t(k)$ does, as shown by the relation derived in Canuto and Dubovikov (1996 **I**, Eq.24):

$$\nu_d(k) = [\nu^2 + \frac{2}{5}\int_k^\infty p^{-2}E(p)dp]^{1/2} = \nu[1 + \mathrm{Re}(k)^2]^{1/2} \tag{3}$$

Here, E(k) is the eddy energy spectrum whose integral over all k's yields the eddy kinetic energy $K = \int_0^\infty E(k)dk$ and where we introduced a k-dependent Reynolds number Re(k). As an example, consider the Kolmogorov spectrum $E(k) = Ko\varepsilon^{2/3}k^{-5/3}$ for fully developed turbulence, Re(k)>1. Eq. (3) then gives:

$$\nu_d \sim \varepsilon^{1/3}k^{-4/3} \sim \varepsilon^{1/3}\ell^{4/3} \tag{4}$$

which is the well-known Richardson's (1926) observationally based "4/3 law". Furthermore, using data from Fig.5 of Callies and Ferrari (2013), one finds $\nu_d \simeq 10^3 m^2 s^{-1}$ which is a commonly used value of the mesoscale diffusivity. Eq.(3) exhibits some interesting features: a) for *small eddies*, k is large, the integral is small and the dynamical viscosity reduces to the kinematic viscosity; b) for *large eddies*, k is small, the integral is large and the kinematic viscosity becomes negligible which corresponds to a turbulent regime, c) that Eq.(3) represents the effect of eddies smaller than k^{-1} is shown by the fact that in the k-integration only eddies >k contribute, d) if observationally based E(k) are available (e.g., Scott and Wang, 2005), Eq.(3) can be used to derive the turbulent viscosity.

Turbulent force $f_i^<$. Though the derivation of this force is more complex, we can offer the following clues. Eq.(1) for a stochastic velocity field is actually never solved as such but only after it is multiplied by another stochastic field and the product averaged so as to yield second-order correlations. An example is the correlation $A_t = \langle \mathbf{f}^< \cdot \mathbf{u}' \rangle$ that enters the equation for the eddy kinetic energy $E(\mathbf{k}) = \langle u_i^2(\mathbf{k}) \rangle /2$ and which physically represents the *work done by the turbulent force* $f_i^<$. To find its expression, we multiply (1) by $u_i(\mathbf{k})$ and average over the directions of **k** using (2) and (3). For the case of homogeneous and isotropic turbulence, we derive the relation:

$$\partial_t E(k) = A_t(k) - 2k^2\nu_d(k)E(k) + A_{ext} \tag{5}$$

where A_{ext} is the work of $f_i(ext)$. Comparing (5) with the standard equation:

$$\partial_t E(k) + 2k^2\nu E(k) = T(k) + A_{ext} \tag{6}$$

the transfer term $T(k)$ is related to the energy flux in k space $\Pi(k)$ as follows:

$$T(k) = -\partial_k \Pi(k) \tag{7}$$

Comparing (6) and (7), we obtain:

$$T(k) = A_t(k) - 2k^2\nu_t(k)E(k), \qquad \nu_t(k) = \nu_d(k)\text{-}\nu \tag{8}$$

Next, we use the locality of the energy transfer in k space, the Kolmogorov hypothesis and the analogy with a flow of water in a pipe in which the mass current is defined as:

$$j(x) = \rho(x)v(x) \tag{9}$$

With the mapping $j \to \Pi, \rho \to E, v \to r$, we may write the turbulence analog of (9) as follows:

$$\Pi(k) = r(k)E(k) \tag{10}$$

where $r(k)$ is the analog of $v(x)$ and may be interpreted as the velocity (rapidity) of the energy flow along the spectrum. Then, from (7), (8) and (10), we derive the relations:

$$T(k) = -\partial_k \Pi(k) = - r\partial_k E(k) - E(k)\partial_k r \tag{11}$$

Comparing with (9), we derive the final relations:

$$\partial_k r = 2k^2\nu_t(k), \qquad r(k) = 2\int_0^k p^2\nu_t(p)dp \tag{12}$$

$$A_t \equiv <\mathbf{f}^< \cdot \mathbf{u}'> = -r\partial_k E(k) \tag{13}$$

which, together with (3), allows one to determine not only the energy spectrum but also other two-points correlation functions as discussed in the 1996-1999 papers.

Application to mesoscales. In *The Physics of Fluids* papers we studied several types of flows. Even though the equations often required a numerical solution, they were much less cumbersome than a numerical simulation of original NSE but still too complex to obtain analytic solutions for the sub-grid functions. For this reason, in Canuto and Dubovikov (2005, CD5) the following approximation was suggested. In the spirit of Prandtl's work (1925), we concentrated on the spectral region in the vicinity of $k \sim k_0$, where k_0 is the location where the energy spectrum E(k) has its maximum. In this region, the correlation function $A_t = <\mathbf{f}^< \cdot \mathbf{u}'>$ vanishes and we may neglect the term $f_i^<$ in Eq.(1). A further assumption was made that the shapes of the spectra of the various correlation functions are similar to that of E(k). Taking into account that in the case of mesoscales there is an inverse energy cascade (e.g., Scott and Wang, 2005), which can be described by a negative dynamical viscosity, we take relation (3) with the negative sign, that is (with $k_0 \sim \ell^{-1}$)

$$\nu(k_0) \sim -K^{1/2}\ell \tag{14}$$

Tracer-thickness fluxes. **Mesoscale diffusivity**. Consider the dynamic equation for a tracer τ :

$$\partial_t \tau + \partial_i(\tau u_i) = S_\tau \tag{15}$$

where the term on the rhs represents small scale turbulent mixing, sources and sinks. Following standard procedure, we write $\tau = \bar{\tau} + \tau'$, $u_i = \bar{u}_i + u_i'$. After substituting into (15) and using the incompressibility condition, one averages the result and obtains the equation for the mean tracer. Subtracting the latter from (15), algebraic steps yield the equation for the τ' field:

$$\partial_t \tau' + \bar{\mathbf{u}} \cdot \nabla_H \tau' + \mathbf{u}' \cdot \nabla_H \bar{\tau} + Q^\tau = 0 \tag{16}$$

where the *non-linear term* Q^τ is defined as follows:

$$Q^\tau \equiv \mathbf{u}' \cdot \nabla_H \tau' - \overline{\mathbf{u}' \cdot \nabla_H \tau'} \tag{17}$$

and where we followed Killworth's (2005) suggestion that because of the strong mixing in the mixed layer, one can use the approximations $\bar{\tau}_z = 0$, $\tau'_z = 0$. Fourier transforming (16) and substituting the non-linear term (17) by its Langevin equivalent analogous to (1), we obtain:

$$\partial_t \tau' = f(\text{ext}) + f^>(\text{turb}) + f^<(\text{turb}) \tag{18}$$

where:

$$f(\text{ext}) \equiv i\mathbf{k} \cdot \bar{\mathbf{u}}\tau' + \mathbf{u}' \cdot \nabla_H \bar{\tau} \quad , \quad f^>(\text{turb}) \equiv -k^2 \chi_t \tau' \tag{19}$$

where χ_t is the turbulent diffusivity which, in Canuto and Dubovikov (1997, **IV,** Eq.7) was expressed in terms of the turbulent viscosity given $\chi_t = \sigma_t^{-1} \nu_t$, where σ_t is the turbulent Prandtl number. Moreover, $f^<(\text{turb})$ is neglected for the reason already discussed. With the further substitution $\partial_t \to -i\omega$, and using the dispersion relation $\omega = \mathbf{k} \cdot \mathbf{u}_d$ derived in Eq.(15a) of CD5, where $\mathbf{u}_d$ is the mesoscale drift velocity discussed by Chelton et al. (2011), Eq.(18) yields:

$$\tau'(\mathbf{k}) = -\frac{\mathbf{u}' \cdot \nabla_H \bar{\tau}}{\chi_t k^2 + i\mathbf{k} \cdot (\bar{\mathbf{u}} - \mathbf{u}_d)}, \qquad |\mathbf{k}| \approx k_0 \tag{20}$$

To construct the horizontal tracer flux $\overline{\mathbf{u}'\tau'}$, one multiplies (20) by $\mathbf{u}'$ and integrates over all k's. In practice, this reduces to considering a kinetic energy spectrum that has a maximum at k_0 where $\partial E(k)/\partial k = 0$ and thus in (18) only $f^>(\text{turb})$ remains. Since $\overline{\mathbf{u}'\mathbf{u}'} \sim K$ (an overbar represents an ensemble average), the tracer flux has the form:

$$\overline{\mathbf{u}'\tau'} = -\kappa_M \nabla_H \bar{\tau} \tag{21}$$

in which, using $k \sim k_0 \sim \ell^{-1}$ and Eq.(3) for ν_d, the mesoscale diffusivity is given by:

$$\kappa_M = \ell K^{1/2} \varpi(\bar{u}, K) \quad , \quad \varpi = [1 + \frac{1}{2K} |\bar{\mathbf{u}} - \mathbf{u}_d|^2]^{-1} \tag{22}$$

In the case of the thickness field, the thickness flux $\overline{\mathbf{u}'h'}$ is derived using a procedure analogous to the one used to derive (21); the bolus velocity is then obtained from $\bar{h}\mathbf{u}^{\dagger} = \overline{\mathbf{u}'h'}$.

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